Caïssan squares: the magic of chess

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In this talk we study various properties of Caïssan magic squares.

An $n \times n$ "magic square" is an array of numbers, usually integers, such that the numbers in all the rows, columns and two main diagonals add up to the same number, the magic sum m. Unless stated otherwise, we assume that $m \neq 0$.

A magic square is Caïssan whenever all paths (with wrap-around) of length n by a chess bishop (and hence by a chess queen) are magic (pandiagonal) and by a (regular) chess knight are magic (CSP2-magic).

Following the seminal 1881 article by "Ursus" [Henry James Kesson (b. c. 1844)] in *The Queen*, we show that 4-pac magic matrices, i.e., 4-ply magic matrices with the "alternate-couplets" property, have rank at most equal to 3.

We also show that an $n \times n$ magic matrix $\mathbf{M}$ with rank 3 and index 1 is EP if and only if $\mathbf{M}^2$ is symmetric. The matrix $\mathbf{M}$ is EP whenever $\mathbf{M}\mathbf{M}^+ = \mathbf{M}^+\mathbf{M}$, where $\mathbf{M}^+$ is the Moore–Penrose inverse.

We identify and study 46080 Caïssan beauties, which are pandiagonal and both CSP2- and CSP3-magic; a CSP3-path is by a special knight that leaps over 3 instead of 2 squares. We find that just 192 of these 46080 Caïssan beauties are EP.

We have tried to illustrate our findings as much as possible, and whenever feasible, with postage stamps or other philatelic items.

Caïssa: "Patron goddess" of chess!

The "patron goddess" of chess players was named Caïssa by Sir William Jones (1746–1794), the English philologist and scholar of ancient India, in a poem entitled "Caïssa" published in 1763.

It seems that the first person to explicitly connect Caïssa with magic squares was "Ursus" [Henry James Kesson, b. c. 1848] in a three-part article entitled "Caïssan magic squares", published in 1881 in *The Queen: The Lady's Newspaper & Court Chronicle.*

In the 8 × 8 Ursus "Caïssan magic square" below, a magic regular-knight's path (CSP2) is marked with red circles and a magic special-knight's path (CSP3) is marked with red boxes (right panel).

A special-knight jumps 3 squares instead of the 2 squares of a regular knight.

28	a	B	c	D	e	F	g	H
61	1	58	3	60	8	63	6	61
52	16	55	14	53	9	50	11	52
45	17	42	19	44	24	47	22	45
36	32	39	30	37	25	34	27	36
5	57	2	59	4	64	7	62	5
12	56	15	54	13	49	10	51	12
21	41	18	43	20	48	23	46	21
28	40	31	38	29	33	26	35	28
61	A	b	C	d	E	f	G	h

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36	32	39	30	37	25	34	27	36
5	57	2	59	4	64	7	62	5
12	56	15	54	13	49	10	51	12
21	41	18	43	20	48	23	46	21
28	40	31	38	29	33	26	35	28
61	A	b	C	d	E	f	G	h

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"THE QUEEN," The Lady's Newspaper, is the oldest and by far the best Lady's Newspaper in existence.

CAISSAN MAGIC SQUARES.

BY "URSUS."

ONCE UPON A TIME, when Orpheus was a little boy, long before the world was blessed with the "Eastern Question," there dwelt in the Balkan forests a charming nymph by name Caïssa. The sweet Caïssa roamed from tree to tree in Dryad meditation fancy free. As for trees, she was, no doubt, most partial

As for trees, Caïssa was, no doubt, most partial to the box and the ebony. *Buxus* is a genus of about 70 species in the family *Buxaceae.* Common names include box (majority of English-speaking countries) or boxwood (North America).

Ebony is a general name for very dense black wood. Some well-known species of ebony include *Diospyros crassiflora* (Gabon ebony), native to western Africa, and *Diospyros celebica* (Makassar ebony), native to Indonesia and prized for its luxuriant, multi-coloured wood grain.

We will denote the "Ursus magic square" (1881) by the "Ursus magic matrix"

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}.$$

The matrix **U** is a classic magic matrix, since the numbers in each row, each column, and in the two main diagonals all sum to the same total, here 260. The magic matrix **U** is classic since it contains the numbers $1, 2, \ldots, 64$ each once.

Moreover, **U** is pandiagonal since the numbers in all diagonals parallel to the two main diagonals of length 8 (with wrap-around) are also magic: the numbers sum to 260.

And so all paths by a queen in chess of length 8 (with wrap-around) are magic.

The 8 row paths, the 8 column paths, the 16 diagonal paths, the 32 paths of type CSP2 (regular knight), and the 16 paths of type CSP3 (special knight), are all magic—the numbers in all these 80 $(= 8 + 8 + 16 + 32 + 16)$ paths, each of length 8 (with wrap-around), all sum to 260.

We will refer to such a matrix as a Caïssan beauty (CB).

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A key purpose in this research is to identify various matrix-theoretic properties of Caïssan beauties.

And to illustrate our findings as much as possible, and whenever feasible with images of postage stamps or other philatelic items. Philatelic beauties!

An important property of $n \times n$ magic matrices involves an $n \times n$ involutory matrix $\mathbf{V}$ that is symmetric, centro-symmetric, and has all row totals equal to 1 and defines an involution in that $\mathbf{V}^2 = \mathbf{I}_n$, the $n \times n$ identity matrix.

The matrix $\mathbf{A}$ is centrosymmetric whenever $\mathbf{A} = \mathbf{FAF}$, where $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ flip matrix:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

which is symmetric, centrosymmetric, and has all row totals equal to 1 and defines an involution in that $\mathbf{F}^2 = \mathbf{I}_n$.

We define an $n \times n$ magic matrix $\mathbf{M}$ with magic sum m to be **V**-associated whenever

$$\mathbf{M} + \mathbf{VMV} = 2m\bar{\mathbf{E}},$$

and **F**-associated whenever

$$\mathbf{M} + \mathbf{FMF} = 2m\bar{\mathbf{E}},$$

where $\mathbf{F}$ is the flip matrix and all the elements of $\bar{\mathbf{E}}$ equal $1/n$.

In an $n \times n$ **F**-associated magic matrix with magic sum m, the sums of pairs of entries diametrically equidistant from the centre are all equal to

$$\frac{2m}{n}.$$

The Euler–Ozanam magic matrix

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 15 & 8 & 10 \\ 12 & 6 & 13 & 3 \\ 14 & 4 & 11 & 5 \\ 7 & 9 & 2 & 16 \end{pmatrix}$$

is **F**-associated, and we find that

$$\frac{2m}{n} = \frac{2 \times 34}{4} = 17.$$

In the literature an **F**-associated magic square is often called (just) "associated" (with no other qualfication) or "regular" or "symmetrical".

For the **F**-associated Euler–Ozanam magic matrix

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 15 & 8 & 10 \\ 12 & 6 & 13 & 3 \\ 14 & 4 & 11 & 5 \\ 7 & 9 & 2 & 16 \end{pmatrix}, \qquad \mathbf{F}\mathbf{M}_1\mathbf{F} = \begin{pmatrix} 16 & 2 & 9 & 7 \\ 5 & 11 & 4 & 14 \\ 3 & 13 & 6 & 12 \\ 10 & 8 & 15 & 1 \end{pmatrix}.$$

Hence $\mathbf{M}_1$ is **F**-associated since $\mathbf{M}_1 + \mathbf{F}\mathbf{M}_1\mathbf{F} = 2m\bar{\mathbf{E}} = 68\bar{\mathbf{E}} = 17\mathbf{E}$, where every element of **E** equals 1.

For the Shortrede–Gwalior magic matrix

$$\mathbf{M}_2 = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix}, \qquad \mathbf{H}\mathbf{M}_2\mathbf{H} = \begin{pmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{pmatrix}.$$

Hence $\mathbf{M}_2$ is **H**-associated since $\mathbf{M}_2 + \mathbf{H}\mathbf{M}_2\mathbf{H} = 2m\bar{\mathbf{E}} = 68\bar{\mathbf{E}} = 17\mathbf{E}$.

For the **F**-associated Euler–Ozanam magic matrix

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The $n \times n$ magic matrix $\mathbf{M}$ with $n = 2h$ even and with magic sum m is **H**-associated whenever

$$\mathbf{M} + \mathbf{HMH} = 2m\bar{\mathbf{E}},$$

where the involutory matrix

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is symmetric, centrosymmetric, and has all row totals equal to 1.

The Shortrede–Gwalior magic matrix

$$\mathbf{M}_2 = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11}^{(2)} & \mathbf{M}_{12}^{(2)} \\ \mathbf{M}_{21}^{(2)} & \mathbf{M}_{22}^{(2)} \end{pmatrix},$$

say, where $\mathbf{M}_{11}^{(2)}, \mathbf{M}_{12}^{(2)}, \mathbf{M}_{21}^{(2)}, \mathbf{M}_{22}^{(2)}$ are all 2×2.

Then $\mathbf{M}_2$ is **H**-associated since

$$\mathbf{M}_{11}^{(2)} + \mathbf{M}_{22}^{(2)} = \mathbf{M}_{12}^{(2)} + \mathbf{M}_{21}^{(2)} = \begin{pmatrix} 17 & 17 \\ 17 & 17 \end{pmatrix}.$$

A key result of our recent research with magic matrices is:

THEOREM. The Moore–Penrose inverse $\mathbf{M}^+$ of the $\mathbf{V}$-associated magic matrix $\mathbf{M}$ is also $\mathbf{V}$-associated, and hence magic.

Here the involutory matrix $\mathbf{V}$ is symmetric, centrosymmetric, and has all row totals equal to 1 and defines an involution in that $\mathbf{V}^2 = \mathbf{I}$.

It follows at once that the Moore–Penrose inverse $\mathbf{M}_1^+$ of the $\mathbf{F}$-associated Euler–Ozanam magic matrix $\mathbf{M}_1$ is $\mathbf{F}$-associated, and the Moore–Penrose inverse $\mathbf{M}_2^+$ of the $\mathbf{H}$-associated Shortrede–Gwalior magic matrix $\mathbf{M}_2$ is $\mathbf{H}$-associated.

Moreover, we find that the Ursus Caïssan magic matrix $\mathbf{U}$ is $\mathbf{H}$-associated and so its Moore–Penrose inverse $\mathbf{U}^+$ is also $\mathbf{H}$-associated.

As we will see, $\mathbf{U}$ has some more interesting properties ...

The Ursus magic matrix **U** has the alternate couplets property (McClintock 1897) in that with **R** denoting the "alternate-couplets summing matrix", we have **RU** =

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix} =$$

$$\begin{pmatrix} 17 & 113 & : & 17 & 113 & : & 17 & 113 & : & 17 & 113 \\ 33 & 97 & : & 33 & 97 & : & 33 & 97 & : & 33 & 97 \\ 49 & 81 & : & 49 & 81 & : & 49 & 81 & : & 49 & 81 \\ 89 & 41 & : & 89 & 41 & : & 89 & 41 & : & 89 & 41 \\ 113 & 17 & : & 113 & 17 & : & 113 & 17 & : & 113 & 17 \\ 97 & 33 & : & 97 & 33 & : & 97 & 33 & : & 97 & 33 \\ 81 & 49 & : & 81 & 49 & : & 81 & 49 & : & 81 & 49 \\ 41 & 89 & : & 41 & 89 & : & 41 & 89 & : & 41 & 89 \end{pmatrix}$$

Emory McClintock (1840–1916)

Charles Planck (1856–1935) studied Caïssan magic squares in 1900 and considered the Caïssan beauty $\mathbf{P} = \mathbf{G}\mathbf{U}$, where $\mathbf{G} = 1 \oplus \mathbf{F}_7$, so that the Planck Caïssan beauty **P** is the Ursus Caïssan beauty **U** with rows $2, 3, \ldots, 8$ reversed.

Planck noted that **P** is "4-ply".

DEFINITION. The $n \times n$ magic matrix **M** with $n = 4k$ doubly-even and magic sum m is 4-ply whenever the four numbers in each of the n^2 subsets of order 2×2 of 4 contiguous numbers (with wrap-around) add up to the same sum $4m/n = m/k$, i.e.,

$$\mathbf{RMR}' = 4m\bar{\mathbf{E}} = \frac{m}{k}\mathbf{E},$$

where **R** is the 8×8 couplets summing matrix.

It follows easily that the Ursus Caïssan beauty **U** is also 4-ply.

We find that $\mathbf{RUR}' =$

$$\mathbf{R}\begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}\mathbf{R}' = \begin{pmatrix} 17 & 113 \\ 33 & 97 \\ 49 & 81 \\ 89 & 41 \\ 113 & 17 \\ 97 & 33 \\ 81 & 49 \\ 41 & 89 \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}(1,1,1,1,1,1,1,1)$$

$$= \begin{pmatrix} 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \\ 130,130,130,130,130,130,130,130 \end{pmatrix}.$$

THEOREM (McClintock 1897). Let **M** denote an $n \times n$ magic matrix with $n = 4k$ doubly-even. Then **M** is 4-ply if and only if it has the alternate couplets property.

DEFINITION. We will say that the $n \times n$ magic matrix **M** with $n = 4k$ doubly-even is 4-pac whenever (with wrap-around) it is 4-ply or equivalently has the alternate couplets property.

We choose the term "4-pac", in part, since "ac" are the initial letters of the two words "alternate couplets".

According to *Wikipedia*
"A sixpack is a set of six canned or bottled drinks, typically soft drink or beer, which are sold as a single unit."

According to Götz Trenkler, in Germany there is the "seven-pack":

> *"Yesterday I had dinner with seven courses!", "Wow, and what did you have?", "Oh, a sixpack and a hamburger!".*

The Ursus Caïssan beauty **U** and the Planck Caïssan beauty **P** are both 4-pac.

We find that

$$\mathbf{U}^2 = \begin{pmatrix} 9570 & 8674 & 9122 & 8226 & 8002 & 7554 & 8450 & 8002 \\ 8674 & 9186 & 8354 & 8866 & 7554 & 8386 & 7874 & 8706 \\ 9122 & 8354 & 8930 & 8162 & 8450 & 7874 & 8642 & 8066 \\ 8226 & 8866 & 8162 & 8802 & 8002 & 8706 & 8066 & 8770 \\ 8002 & 7554 & 8450 & 8002 & 9570 & 8674 & 9122 & 8226 \\ 7554 & 8386 & 7874 & 8706 & 8674 & 9186 & 8354 & 8866 \\ 8450 & 7874 & 8642 & 8066 & 9122 & 8354 & 8930 & 8162 \\ 8002 & 8706 & 8066 & 8770 & 8226 & 8866 & 8162 & 8802 \end{pmatrix}$$

is 2×2 block-Latin with 4×4 blocks.

Indeed, if the magic matrix $\mathbf{M}$ is $\mathbf{H}$-associated, then $\mathbf{M}^2$ is block-Latin.

THEOREM. Suppose that the $n \times n$ magic matrix $\mathbf{M}$ with magic sum m is $\mathbf{H}$-associated with $n = 2h$ even. Then $\mathbf{M}^2$ and $\mathbf{MHM}$ are block-Latin, i.e.,

$$\mathbf{M}^2 = \begin{pmatrix} \mathbf{K}_1 & \mathbf{L}_1 \\ \mathbf{L}_1 & \mathbf{K}_1 \end{pmatrix}, \quad \mathbf{MHM} = \begin{pmatrix} \mathbf{K}_2 & \mathbf{L}_2 \\ \mathbf{L}_2 & \mathbf{K}_2 \end{pmatrix},$$

for some $h \times h$ matrices $\mathbf{K}_1, \mathbf{K}_2, \mathbf{L}_1, \mathbf{L}_2$. Moreover,

$$\mathbf{K}_1 + \mathbf{L}_2 = \mathbf{K}_2 + \mathbf{L}_1 = m^2 \bar{\mathbf{E}}_h,$$

where $\bar{\mathbf{E}}_h$ is the $h \times h$ matrix with all entries equal to $1/h$, with $h = n/2$.

DEFINITION. We define the $n \times n$ magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ to be keyed whenever its characteristic polynomial is of the form

$$\det(\lambda\mathbf{I} - \mathbf{M}) = \lambda^{n-3}(\lambda - m)(\lambda^2 - \kappa),$$

where the magic key

$$\kappa = \tfrac{1}{2}\left(\mathrm{tr}\mathbf{M}^2 - m^2\right)$$

may be positive, negative or zero. When $\kappa \neq 0$ then $\mathrm{rank}(\mathbf{M}) = 3$.

An $n \times n$ keyed magic matrix with magic sum m, therefore, has eigenvalues m and $\pm\sqrt{\kappa}$ in addition to $n - 3$ necessarily 0 eigenvalues.

DEFINITION. The $n \times n$ matrix $\mathbf{A}$ has index 1 whenever $\mathrm{rank}(\mathbf{A}^2) = \mathrm{rank}(\mathbf{A})$.

The Ursus Caïssan beauty $\mathbf{U}$ and the Planck Caïssan beauty $\mathbf{P}$ both have rank 3 and index 1 and both are keyed. The magic keys are $\kappa(\mathbf{U}) = 2688$ and $\kappa(\mathbf{P}) = 960$.

THEOREM. Let the magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ be keyed with magic key $\kappa \neq 0$ and index 1.

Then $\mathbf{M}$ has rank 3 and all odd powers are "linear in the parent" in that

$$\mathbf{M}^{2p+1} = \kappa^p \mathbf{M} + m(m^{2p} - \kappa^p)\bar{\mathbf{E}} \quad (1)$$

for $p = 1, 2, 3, \ldots$

Moreover, the group inverse

$$\mathbf{M}^{\#} = \frac{1}{\kappa}\mathbf{M} + m\left(\frac{1}{m^2} - \frac{1}{\kappa}\right)\bar{\mathbf{E}} \quad (2)$$

is also "linear in the parent".

The right-hand side of (2) is the right-hand side of (1) with $p = -1$.

DEFINITION. The $n \times n$ index-1 matrix $\mathbf{A}$ has a group inverse $\mathbf{A}^{\#}$ which satisfies the three conditions

$$\mathbf{A}\mathbf{A}^{\#}\mathbf{A} = \mathbf{A}$$

$$\mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}$$

$$\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}\mathbf{A}.$$

DEFINITION. We define a square matrix to be EP whenever the "Equal Projectors" property

$$\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+\mathbf{A}$$

holds. Here $\mathbf{A}^+$ is the Moore–Penrose inverse of $\mathbf{A}$.

BEAUTIFUL THEOREM (new?). Let the magic matrix $\mathbf{M}$ have magic sum $m \neq 0$, rank 3 and index 1.

Then $\mathbf{M}$ is EP if and only if $\mathbf{M}^2$ is symmetric.

For the classic Shortrede–Gwalior magic matrix $\mathbf{M}_2$ with magic sum $m = 34 \neq 0$:

$$\mathbf{M}_2 = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix},$$

we find

$$\mathbf{M}_2^2 = \begin{pmatrix} 345 & 281 & 273 & 257 \\ 281 & 313 & 257 & 305 \\ 273 & 257 & 345 & 281 \\ 257 & 305 & 281 & 313 \end{pmatrix}$$

is symmetric, and since $\mathbf{M}_2$ has rank 3 and index 1 it is EP.

Moreover, $\mathbf{M}_2^2$ is 2×2 block-Latin with symmetric 2×2 blocks, and so $\mathbf{M}_2^2$ has 6 distinct entries.

Philatelic Block-Latin Square (PBLS) with
non-symmetric 2×2 blocks featuring 8 stamps for “Pioneers of Flight”:
(top left block) James H. Doolittle, Claude Dornier, Ira C. Eaker, Jacob C. H. Ellehammer;
(top right block) Henry H. Arnold, Louis Blériot, William E. Boeing, Sydney Camm;

Micronesia 1996, *Scott* 238.

We find that

$$\mathbf{U}^2 = \begin{pmatrix} 9570 & 8674 & 9122 & 8226 & 8002 & 7554 & 8450 & 8002 \\ 8674 & 9186 & 8354 & 8866 & 7554 & 8386 & 7874 & 8706 \\ 9122 & 8354 & 8930 & 8162 & 8450 & 7874 & 8642 & 8066 \\ 8226 & 8866 & 8162 & 8802 & 8002 & 8706 & 8066 & 8770 \\ 8002 & 7554 & 8450 & 8002 & 9570 & 8674 & 9122 & 8226 \\ 7554 & 8386 & 7874 & 8706 & 8674 & 9186 & 8354 & 8866 \\ 8450 & 7874 & 8642 & 8066 & 9122 & 8354 & 8930 & 8162 \\ 8002 & 8706 & 8066 & 8770 & 8226 & 8866 & 8162 & 8802 \end{pmatrix}$$

is symmetric and so the Ursus Caïssan beauty **U** is EP.

And hence $\mathbf{U}^{\#} = \mathbf{U}^{+}$, which is also a Caïssan beauty.

Moreover, $\mathbf{U}^2$ is 2×2 block-Latin with 4×4 blocks.

The oldest EP classic 4×4 magic square may be the one discovered in 1841 by Robert Shortrede (1800–1866), dated 1483.

We define it by the "Shortrede–Gwalior magic matrix"

$$\mathbf{M}_2 = \begin{pmatrix} 16 & 9 & 4 & 5 \\ 3 & 6 & 15 & 10 \\ 13 & 12 & 1 & 8 \\ 2 & 7 & 14 & 11 \end{pmatrix},$$

which is EP, as well as 4-pac, **H**-associated and pandiagonal, rank 3 and index 1.

The 1842 article, which announced the discovery of the Shortrede–Gwalior magic square, is signed by "Captain Shortreede", who we believe was (later) Major-General Robert Shortrede (1800–1868), with the extra "e" in "Shortreede" here a typo.

The magic square defined by $\mathbf{M}_2$ was discovered in 1841 by Robert Shortrede in an (unidentified) old temple in Gwalior (Madhya Pradesh, about 120 km south of Agra).

१६	९	४	५
३	६	१५	१०
१३	१२	१	८
२	७	१४	११

The original Shortrede–Gwalior magic square with entries in Sanskrit.

The fort at Gwalior: India 1984, Scott 1065.

Robert Shortrede went to India in 1822 and was appointed to the Great Trigonometric Survey (GTS) in which he remained until 1845.

The GTS was piloted in its later stages by Sir George Everest (1790–1866).

GTS: India 2004, Scott 2067a

25th anniversary of the coronation of Queen Elizabeth II on 2 June 1953:

Great Britain 1978, Scott 835–838

Shortrede observed that the magic square defined by

$$\mathbf{M}_2 = \begin{pmatrix} 16 & 9 & \mathbf{4} & 5 \\ \mathbf{3} & 6 & 15 & 10 \\ 13 & 12 & \mathbf{1} & 8 \\ \mathbf{2} & 7 & 14 & 11 \end{pmatrix}$$

has a "rhomboid" property— the places of the numbers $1, 2, 3, 4$ (bold-face red) in $\mathbf{M}_2$, form a "rhomboid", as do the numbers $5, 6, 7, 8$; $9, 10, 11, 12$ and $13, 14, 15, 16$.

We will say that a magic matrix with such a rhomboid property is "rhomboidal".

In *Wikipedia* a "rhomboid" is defined as a parallelogram in which adjacent sides are of unequal lengths and the angles are all oblique (not equal to 90°).

A parallelogram with sides of equal length is a "rhombus"; a parallelogram with right-angled corners is a "rectangle" and a rectangle with all sides of equal length is a "square".

This stamp, which the *Scott Catalogue* (#851) calls "Rhomboid with yellow lines", features two "rhomboids" which are actually squares each tilted at 45°.

The Dutch *Postzegelcatalogus* identifies the stamp as "Compositie met gele lijnen" (Composition with yellow lines).

The 1933 painting by Piet Mondrian (1872–1944): "Ruitvormige compositie met vier gele lijnen" [Lozenge composition with four yellow lines] is shown on the left of the stamp.

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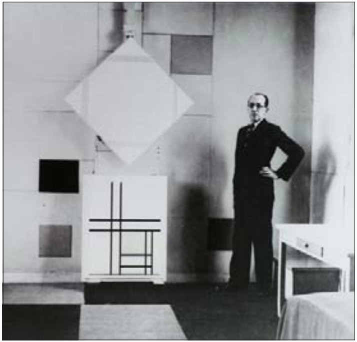

We believe that the stamp is based on a photograph by Charles Karsten of Piet Mondrian with his 1933 painting "Ruitvormige compositie met vier gele lijnen" taken in Mondrian's atelier in Paris at 26, rue du Départ (near Gare Montparnasse), c. October 1933.

"The other day I noticed that I can no longer pose for a picture without putting my hand on my hip. It's supposed to make you look thinner, but really it makes everyone look like a little teapot!"–Jezebel

Suur aitäh: Imbi Traat (Tartu 2000).

We now present a matrix representation of a generalized version of the algorithm for a (classic) pandiagonal magic square given for a special case in the book

Recreations with Magic Squares: the eight queens' problem solved by magic squares and domino squares,

by "Cavendish", pub. Thomas De La Rue, London, 1894.

"Cavendish" is the *nom de plume* of Henry Jones (1831–1899), an English author well-known as a writer and authority on tennis and card games, especially whist.

"Cavendish" (1894) defines a Caïssan magic square as (being just) pandiagonal, but the algorithm he gives (for just a single special case) yields a Caïssan beauty that has all CSP2- and CSP3-paths magic as well as being pandiagonal.

Our generalized Cavendish matrices $\mathbf{C}_{\mathbf{s},t,u}$ are also Caïssan beauties, and like the Ursus matrix $\mathbf{U}$, are also 4-pac, $\mathbf{H}$-associated, and keyed with rank 3 and index 1.

Moreover, the $\mathbf{C}_{\mathbf{s},t,u}$ are EP for all values of the "seed parameters" $\mathbf{s}$, t, u.

We define the 8×8 matrix

$$\mathbf{B}_{\mathbf{s},t} = \begin{pmatrix} a & t-a & a & t-a & a & t-a & a & t-a \\ b & t-b & b & t-b & b & t-b & b & t-b \\ c & t-c & c & t-c & c & t-c & c & t-c \\ d & t-d & d & t-d & d & t-d & d & t-d \\ t-a & a & t-a & a & t-a & a & t-a & a \\ t-b & b & t-b & b & t-b & b & t-b & b \\ t-c & c & t-c & c & t-c & c & t-c & c \\ t-d & d & t-d & d & t-d & d & t-d & d \end{pmatrix},$$

and the 8×8 "Cavendish" magic matrix with seed vector $\mathbf{s} = (a, b, c, d)'$ and seed scalars t, u

$$\mathbf{C}_{(a,b,c,d)',t,u} = \mathbf{C}_{\mathbf{s},t,u} = \mathbf{B}_{s,t} + 8\mathbf{B}'_{s,t} - u\mathbf{E}_8 =$$

$$\begin{pmatrix} 9a-u & t-a+8b-u & a+8c-u & t-a+8d-u & -7a+8t-u & 9t-a-8b-u & a+8t-8c-u & 9t-a-8d-u \\ b+8t-8a-u & 9t-9b-u & b+8t-8c-u & 9t-b-8d-u & b+8a-u & t+7b-u & b+8c-u & t-b+8d-u \\ c+8a-u & t-c+8b-u & 9c-u & t-c+8d-u & c+8t-8a-u & 9t-c-8b-u & -7c+8t-u & 9t-c-8d-u \\ d+8t-8a-u & 9t-d-8b-u & d+8t-8c-u & 9t-9d-u & d+8a-u & t-d+8b-u & d+8c-u & t+7d-u \\ t+7a-u & a+8b-u & t-a+8c-u & a+8d-u & 9t-9a-u & a+8t-8b-u & 9t-a-8c-u & a+8t-8d-u \\ 9t-b-8a-u & -7b+8t-u & 9t-b-8c-u & b+8t-8d-u & t-b+8a-u & 9b-u & t-b+8c-u & b+8d-u \\ t-c+8a-u & c+8b-u & t+7c-u & c+8d-u & 9t-c-8a-u & c+8t-8b-u & 9t-9c-u & c+8t-8d-u \\ 9t-d-8a-u & d+8t-8b-u & 9t-d-8c-u & -7d+8t-u & t-d+8a-u & d+8b-u & t-d+8c-u & 9d-u \end{pmatrix},$$

where all the entries of the 8×8 matrix $\mathbf{E}_8$ are equal to 1.

The Cavendish matrix $\mathbf{C}_{\mathbf{s},t,u}$ has rank 3 and index 1.

It is then easy to see that $\mathbf{C}_{\mathbf{s},t,u}$ is EP.

We recall that

$$\mathbf{C}_{\mathbf{s},t,u} = \mathbf{B}_{s,t} + 8\mathbf{B}'_{s,t} - u\mathbf{E}_8.$$

Then $\mathbf{C}_{\mathbf{s},t,u}$ is EP since

$$\mathbf{C}^2_{\mathbf{s},t,u} = 8(\mathbf{B}_{\mathbf{s},t}\mathbf{B}'_{\mathbf{s},t} + \mathbf{B}'_{\mathbf{s},t}\mathbf{B}_{\mathbf{s},t}) + (130t^2 - 72tu + 8u^2)\mathbf{E}_8$$

is (clearly) symmetric.

In 1869, Henry Jones "Cavendish" joined the All England Croquet Club, and in 1875, he proposed that one of the club's croquet lawns be set aside for the playing of lawn tennis. In 1877, it was proposed that a lawn tennis championship be held, and "The Championships" were born.

"The Championships, Wimbledon", or simply "Wimbledon" is the oldest tennis tournament in the world and is considered by many to be the most prestigious.

Martina Navratilova (b. 1956) & Boris Becker (b. 1967)

Michael Stich (b. 1968) & Steffi Graf (b. 1969).

"Jones, how about a moratorium on keeping up? Our budget's a bit tight this year."

We conclude by asking if this "Cavendish" = Henry Jones (1831–1899), the English author well-known as a writer and authority on tennis and card games, is a descendant of Sir William Jones (1746–1794), the English philologist and scholar of ancient India, who identified chess with the goddess Caïssa in his 1763 poem.

"Keeping up with the Joneses" is an idiom (in the English-speaking world) for the comparison to one's neighbour as a benchmark for the accumulation of material goods. The phrase was popularized with a comic strip created by cartoonist Arthur R. "Pop" Momand (1886–1987) in 1913.
